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Mathematical Programming: an aid to executive decisions.

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MATHEMATICAL PROGRAMMING:
AN AID TO EXECUTIVE DECISIONS

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PREFACE

The road from inception to use of a new technique in management has frequently been a rough and tortuous one. This fact is due in part to natural human reluctance, even in a nation which prides itself on its progressivism in business, to meddle with what is, or is believed to be, a smoothly running machine. But it is often due primarily to the frequent mysticism of figures which surrounds a new technique in modern times; the arcana with which the theorist shrouds his discoveries; the esoteric technical dissertation replete with alphas and betas that render it incomprehensible to the non-mathematical layman.

The aim of this paper, which explores the subject of Mathematical Programming, is fourfold:

First: to cite enough examples to prove the case that Mathematical Programming is really a technique for modern business and military management.

Second: to reduce the many highly technical and theoretical references on the subject to a form that can be readily comprehended by personnel at the executive level who may not have a background of mathematics; and whenever possible to remove completely from the text any formulae, numbers, or mystifying symbols.

Third: to show that the opportunities for applying this management tool are not confined to any particular business, but are, within certain limitations, widespread in their fields of application.

Fourth: to give directions for computation by two of the simplest methods of solution of common type problems. In the interests of the second aim, these computational methods have been relegated to appendices; the readers who have a mental block against figures of any sort will not find them impeding their perusal of the text.

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INTRODUCTION

The Executive has ever been faced with the problemsof making decisions. He may make them based on his own experience as it relates to the current question posed; on the authority of some other person because of the latter's position or technical proficiency; on an array of factual material; or on the basis of beliefs or feelings - that is to say, on intuition.[1] Any tool which he may use to assist him in the decision-making process, particularly one which enables him to make a scientific and quantitative analysis, should be most welcome to him.

The scientist would be the first to agree that quantitative aspects are not the whole story for decision-making. "Many other aspects can and do enter; politics, morale, tradition - items important but not numerically expressed. The prerogative and indeed the responsibility of the executive is to add these factors to the quantitative basis to reach a final decision."[2] But the decision-maker is on far firmer ground when he has full and accurate knowledge of the quantitative aspects that do exist; he can then if he so decides depart from the course that those aspects dictate as optimum with the assurance of his awareness as to the effect of that departure. One of the tools which has within recent years made remarkable strides in the field of quantitative analysis of business and military problems is mathematical, or linear programming.

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[1]Richard N. Owens, Introduction to Business Policy, (Homewood, Illinois; Richard D. Irvin, Inc., 1954), pp.118-20.

[2]Phillip M. Morse and George E. Kimball, Methods of Operations Research, (Cambridge, Mass.: Technology Press of the Massachusetts Institute of Technology and John Wiley, New York, jointly, 1951), p.l.

CHAPTER I

BASIC PRINCIPLES

Mathematical programming is, essentially, the selection by rational means of an optimum course of action from among a set of courses of action.[1] The general nature of many problems facing management[2] is one in which "a group of limited resources must be shared among a number of competing demands, and all decisions are 'interlocking' because they all have to be made under the common set of fixed limits."[3] Typical among such problems are determination of the following:

1. The most efficient routing (in terms of transportation costs, or time, or requirements for use of ships or aircraft) of material from a group of supply or production points to a group of storage or user points.

2. The efficient and economical use of scarce raw materials, or machines, to produce the maximum output of critical goods.

3. The best assignment of output production of military items, such as planes, between competing demands of combat and training missions.

4. The minimum cost of contract awards to meet a given supply requirement from among a group of bidders quoting various prices, maxima and minima

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[1]Joseph O. Harrison, Jr., "Linear Programming and Operations Research," Operations Research for Management, ed. Joseph F. McCloskey and Florence N. Trefethen (Baltimore: The Johns Hopkins Press, 1954), p. 237.

[2]Such terms as management, executive, etc., are used in the broad sense, and apply equally to military, governmental, and business organizations.

[3]Alexander Henderson and Robert Schlaifer, "Mathematical Programming," Harvard Business Review, XXXII(May-June, 1954), p. 74.

production for various items, and other quantity or price restrictions as to bids.

As can be readily seen from the above examples, the limited resources may be manpower, or capital, or material, or equipment, or more probably some combination of the four.

Mathematical programming has also been termed a restatement of the central formal problem of economics - that of allocating scarce resources so as to maximize the attainment of some predetermined objective - in a form which is designed to be useful in making practical decisions in business and economic affairs.[4] The accent here is on the word "practical." The theory and principles that have led to the development of the mathematical programming techniques coming increasingly into business and military use may well be far beyond the comprehension of the non-mathematical layman, but there is nothing theoretical about their applications; a glance at the four typical problems listed above should dispell any doubt on that score. "It is no longer premature to say that mathematical programming has proved its worth . . . for finding optimum economic programs . . . . It is a practical tool for business planning."[5]

But this new technique is more than just a new method for solving problems that have been with us for many years, and which have been solved before the technique came on the scene. The solutions it offers are given much faster, more easily, and more accurately than ever before.[6] Management is assured that the solutions made are the best possible,and hence need spend

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[4]R. Dorfman, "Mathematical, or 'Linear' Programming," American Economic Review, XLIII (December, 1953), p.797.

[5]Ibid., p. 821.

[6]Henderson and Schlaifer, p. 73.

no further time on fruitless attempt at improvement, but may concentrate on production or execution of the task, and on other planning matters. Higher echelon management is spared time-consuming (and, at its salary level, expensive) effort on solving problems that high grade clerical personnel can solve, given the necessary instructions as to how to proceed in any given case. For repetitive programming work, the advantages are obvious - and tremendous. And for more extensive (though not necessarily more complicated) problems, with greater number of variables, mechanical or electronic computers using mathematical programming techniques offer answers to problems such as cost or profit implications of proposed series of changes that management would never have the time to analyze and solve whether it wanted to or not.

A simple example of an application of mathematical programming is probably the best manner in which to explain and demonstrate just what it can do.[7] Suppose that a regional freight car distributor must send empties from three divisions on which car surpluses are predicted, to five other divisions on which shortages are expected. Further assume the total shortages are exactly equal to the total surpluses, though this is not a limitation for other than simplifying the example. Conditions can best be expressed in a simple table:

Origin	Destination					Surpluses
	D1	D2	D3	D4	D5	
S1						9
S2						4
S3						8
Shortages	3	5	4	6	3	21

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[7]A word of caution is necessary here. Most examples given in the many references on this subject have necessarily been extremely simple; so simple, in fact, as to be susceptible of solution by trial and error if not by inspection. But if the examples were too extensive, they would fail as instructive and readily comprehendible examples.

The table merely states that Division S1 has a surplus of 9 cars; D2 a shortage of 5 cars, etcetera. The cost of shipping an empty car from a shipping point to a destination is tabled as follows:

Origin	Destination				
	D1	D2	D3	D4	D5
S1	10	20	5	9	10
S2	2	10	8	30	6
S3	1	20	7	10	4

The table indicates that it costs $10 to send an empty car from Division S1 to D1; $30 to send a car from S2 to D4, etcetera. Now, while it is possible to obtain a feasible solution by inspection, it is not possible to know whether the solution selected is the least cost solution, or what greater cost, if any, the solution has than the least cost. Mathematical programming offers a method not only of obtaining the least cost solution, but also of ensuring that it is in fact the least cost. Appendix "A" gives the solution to this problem. The minimum cost is $150; one might find it interesting to make up a feasible solution, and cost it to see how it compares with the $150 figure.[8]

When one realizes that the same technique is capable of solving problems involving a table 300 x 500 as easily as the 3 x 5 table shown in the example, the advantages are obvious. The time required is of course greater, but the technique is identical, and with the aid of electronic computers the time required is a matter of hours or minutes.

A somewhat more extensive example of the same type, and solved by the same procedure - which is described in step-by-step detail in Appendix "A" - is a case where the technique is currently in use as a routine operating

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[8] "Operational Research in Distributing Empty Cars," Railway Age, CXXXIV(April 20, 1953), pp. 73-4.

procedure in an actual company. It is as follows:

> The H. J. Heinz Company manufactures ketchup in half a dozen plants scattered across the United States from New Jersey to California and distributes this ketchup from about 70 warehouses located in all parts of the country.
>
> In 1953 the company was in the fortunate position of being able to sell all it could produce, and supplies were allocated to warehouses in a total amount exactly equal to the total capacity of the plants. Management wished to supply these requirements at the lowest possible cost of freight; speed of shipment was not important. However, capacity in the West exceeded requirements in that part of the country, while the reverse was true in the East; for this reason a considerable tonnage had to be shipped from western plants to the East. In other words, the cost of freight could not be minimized by simply supplying each warehouse from the nearest plant.[9]

The Heinz problem was solved, and for the first time, in about twelve hours by one man with no other equipment than a pencil and paper. Subsequently clerks, trained in the routine for this problem, solved it in the normal course of business in considerably less time.

One may have the impression from the previous examples that mathematical programming is limited to problems involving shipping or transportation. Nothing could be further from the truth. Chapter II indicates the varied types of problems currently being solved in business and industry, and Chapter III some current military applications. Even at the present early stage of application of this new decision-making aid, its uses have been manifest in striking fashion in many completely unrelated fields.

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[9]Henderson and Schlaifer, p. 77.

CHAPTER II

USE IN BUSINESS AND INDUSTRY

A type of problem which requires the combination of various input items into output items in the most efficient manner possible according to some specified criterion is often called a mixing problem.[1] One such problem has been analyzed and solved by G. F. Stigler[2] in which he considered the determination of an adequate diet of minimum cost. Admittedly, such aesthic considerations as palatability and variety were omitted, and certain simplifying assumptions were made. The problem is this: Given certain specific foods, their costs, and their nutrient content, what is the combination of foods and their amounts which are of least cost in a stated time period. Stigler worked out the problem for 77 different foods and 9 different basic nutrients, for 1939 and 1944 prices. He used nutrients of calories, protein, calcium, iron, Vitamin A, thiamine, riboflavin, niacin, and ascorbic acid. He found that in 1939 the optimum diet for an active 155 lb. man consisted largely of wheat flour, cabbage, and dried navy beans, with a total cost of $39.93 for the year. With 1944 prices, the optimum solution required a change of navy beans to hog liver, and the cost of the diet was $59.88 for that year. The exact quantities and costs are shown in the table below:

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[1]Harrison, p. 228.

[2]George F. Stigler, "The Minimum Cost of Subsistence," <u>Journal of Farm Economics</u>, XXVII (May, 1945), pp. 303-14.

	August 1939		August 1944	
Wheat flour	370 lbs.	$13.33	535 lbs.	$34.53
Evaporated milk	57 cans	3.84	-	-
Cabbage	111 lbs.	4.11	107 lbs.	5.23
Spinach	23 lbs.	1.85	13 lbs.	1.56
Dried Navy beans	285 lbs.	16.80	--	--
Pancake flour	--	--	134 lbs.	13.08
Pork liver	--	--	25 lbs.	5.48
		$39.93		$59.88

This problem may not appear of much practical interest, since certainly no one is going to eat only beans and cabbage all year long, but if the method is applied to cattle feed or hog feed it appears more attractive; and if used on a large farm the savings might well be considerable. F. V. Waugh's "The Minimum Cost of Dairy Feed"[3] uses the linear programming method in a determination of the minimum cost of feeding a dairy herd; and a similar article on minimum feed costs for hogs, together with certain philosophical discussion of the food production problem as a national strategy in time of war or emergency, was published by Christensen and Mighell.[4]

One may well ask how Stigler was able to apply the mathematical procedures in 1945 that were not generally developed and published before 1947. In truth, Stigler did not use strict mathematical programming to obtain his solution, and is careful to point out that it might be possible to better his answer; he used a logical, trial-and-error approach to solve his problem, which is of course exactly suited to the linear programming approach of the

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[3]Frederick V. Waugh, "The Minimum Cost of Dairy Feed," Journal of Farm Economics, XXXIII(August, 1951), pp. 299-310.

[4]R. P. Christensen and R. L. Mighell, "Food Production Strategy and the Protein-Feed Balance," Journal of Farm Economics, XXXIII (August, 1951).

Transportation-Procedure, to be covered more fully in Chapter IV. Stigler also admits quite readily that a wider variety of foods would offer a decreased cost; and that he learned of a type of diet used for laboratory rats that if expanded in amount to feed a man would cost only $27 a year based on 1939 prices. His main point, however, is that diets which are stated as being minimum cost diets by various writers actually cost two to three times as much as his diet, the extra cost being that associated with the aesthetic and taste factors which he ignored. One may be willing to pay for his tastes, but should know how much he is paying.

A very similar problem occurs in the gasoline industry in connection with the blending of straight-run gasolines received from the refinery with other products such as isopentane, and chemical catalysts such as tetraethyl lead. The resulting gasolines are graded on a performance basis, and the two most common and important rating factors are those of octane ratings and volatility. This problem had been set up as a mathematical programming problem with ten constraints, or requirements to be met, and twenty-two unknowns, and an optimal solution obtained.[5] A particularly interesting part of the solution was the determination that a company policy, requiring for trade reasons that a small amount - 500 barrels daily - of a particular aviation gasoline be made, even though it was known that it was somewhat more costly to do so, actually was costing the company $80,000 annually in profits. The company was then in a position to decide whether the policy was to be continued or not, and knew when the decision was made just what the cost of its policy was.

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[5]A. Charnes, W. W. Cooper, and B. Mellon, "Blending Aviation Gasolines," Symposium on Linear Inequalities and Programming, (Washington, D. C.: Planning Research Division, Directorate of Management Analysis, DCS/Comptroller, Head-Quarter, USAF, April, 1952), pp. 115-46. Reprinted in Econometrica, XX, (April, 1952), pp. 135-59.

The foregoing problems are similar, but not identical. In the first case, we were given a prescribed quantity of each input (iron, calcium, etc.) and a prescribed lower limit of value of each input (cost of the various foods considered) and were to find the number of units of each which would minimize the value of the total input. In the second case, we were given a prescribed unit for each output (price of a given gasoline mixture) and a prescribed upper limit for each input (amounts of tetraethyl lead, etc.) and were to determine the units of output that should be produced in order to maximize the value of the total output. Stated more simply, the first is a valuation problem, and the second a production scheduling problem. These two problems are obviously closely related, and are known as duals of each other.[6] Dual problems have a number of important properties involving abstract mathematical considerations; but the physical counterparts of the mathematical variables do exist, and consideration of a problem of the mixture type may well require consideration of its dual to obtain greater insight into its intrinsic value. As will later be indicated methods of solution of types of mathematical programming are in some instances based on duality properties.

A different problem - the one used in Chapter I as an example - is the type generally referred to as a transportation problem. This problem was formulated independently by Hitchcock in 1941[7] and Koopmans in 1947[8] and is consequently often called the Hitchcock-Koopmans Transportation Problem. Certainly the class of transportation-type problem applications is quite numerous, and at the present state of the science is probably the most fruitful

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[6]Harrison, p. 228.

[7]F. L. Hitchcock, "The Distribution of a Product from Several Sources to Numerous Localities," Journal of Mathematics and Physics, XX(1941), pp. 224-30.

[8]T. C. Koopmans, "Optimum Utilization of the Transportation System," Proceedings of the International Statistical Conferences, 1947, V. Reprinted as Supplement to Econometrica, XVII (July 1949), pp. 136-45.

in the linear programmer's repertoire.[9] Most simply stated, it is as follows:

> A homogeneous product is available in specified amounts at each of a number of different origins. One is to ship a specified amount of the product to each of a number of different destinations, and the cost to make a shipment of one unit of product is known. What is the minimum cost of a feasible shipping schedule?

A broader definition, and one probably more appropriate for this type of problem, is the term "assignment problem."[10] One such problem, actually so termed, is that of personnel assignment.[11] This is a problem of fitting a given number of persons into a specified but different number of personnel or job categories, in which the capabilities of any one person in each of the job categories is known by measurement tests, or scores, and the object sought is to make a maximum of the overall productivity of the group. To give some concept of the size of this job, an electronic computer such as the UNIVAC, one of the most advanced types of computers at present, solves problems up to eight jobs and fifty types of people in an hour and a half.

That mathematical programming has application in industry for a manufacturer's scheduling type of problem is shown by its use by SKF Industries in connection with machine scheduling. Given the cost to operate a given machine, and various orders which can be filled on various machines, SKF uses the technique to determine the least cost to produce the orders on hand. Weekly revisions of the schedule are made, with due allowance for the work in progress, and a new schedule for the week drawn up, with such rescheduling of work in progress as is necessary. SKF has indicated savings on an annual basis of this system of $100,000.[12]

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[9]Walter Jacobs, "Military Applications of Linear Programming," Proceedings of the Second Symposium in Linear Programming, (Washington, D.C.: Directorate of Management Analysis, DCS/Comptroller, Headquarters, USAF, 1955), I, p. 10.

[10]Harrison, p. 229.

[11]D. F. Votaw, Jr., and A. Orden, "The Personnel Assignment Problem," Symposium on Linear Inequalities . . ., pp. 155-63.

[12]"New Machine Loading Methods," Factory Management Maintenance, CXII (January, 1954), pp. 136-7.

An extension of the Heinz type of problem can occur when production capacity exceeds warehouse or other demand requirements, and the cost of production varies from one plant to another. Least total cost rather than merely least shipping cost is therefore of importance, and it is as important to produce in the right place as to ship the right amount to the right place. Though it is tempting to decide each of these problems as a separate problem, such a method will not generally give the least total cost.[13] A solution to this type of problem may also very possibly give management considerably more information than just the cheapest way to do a particular job. It may well turn out that the least cost solution requires running one plant at a very low per cent of rated output. Management well might ask whether that plant should not be closed entirely, and the small output taken care of by overtime at the other plants. Many factors other than total cost will enter into the final decision, but the decision-maker will know what the costs of various solutions to his problem are. An excellent example of this type of problem, simple and easy to follow, is given by Henderson and Schlaifer.[14]

One may have inferred that mathematical programming can handle only static problems. Such inference is incorrect. Problems in which a time factor is involved - and such occurs in many practical problems - can in many cases be handled by mathematical programming procedures. In producing any commodity, particularly one with seasonal demand, there is always the conflict between (a) running steady production runs and letting storage take care of

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[13]Henderson and Schlaifer, pp. 79-80.
[14]Ibid., p. 81.

the fluctuations in inventory caused by varying sales, and (b) keeping inventory at a constant level and letting production take care of the variation. Most probably there is a happy medium, and management would undoubtedly like to find it. Applications of this problem in business and military life are many; J. F. Magee has covered this subject in some detail as regards business production scheduling.[15] Military applications are discussed in Chapter III. A variation of this problem is one in which the amount demanded is uncertain but has a probability distribution.[16]

An example of a problem in which the time factor is one of the variables is the so-called caterer's problem.[17] This problem is a paraphrased version of a practical military problem which arose in connection with the estimation of aircraft spare engine requirements; as is so often the case with practical examples of mathematical programming problems, it is difficult to tell whether the military application leads to the civil application or vice versa. The problem is as follows:

> A caterer knows that in connection with the meals he has arranged to serve the next stated number of days he will need a specific number of napkins on each particular future day; this number may vary from day to day. Laundering takes a fixed number of days at a fixed price on the standard service, or a lesser number of days, which is known, at a more costly price on the special service. Starting with no napkins on hand or in the laundry, how many napkins shall the caterer buy at a known price, and how shall he proceed to meet his needs and minimize his outlays for the stated number of days?

It is hoped that the foregoing will give an idea of the scope of problems for which mathematical programming can offer assistance of considerable

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[15]John F. Magee, Studies in Operations Research, I: Linear Programming in Production Scheduling, (Cambridge, Mass.: Arthur D. Little, Inc., n.d.).

[16]Gunnar Dannerstedt, "Production Scheduling for an Arbitrary Number of Periods Given the Sales Forecast in the Form of a Probability Distribution," Journal of Operations Research Society of America, III (August, 1955), pp. 300-18.

[17]W. W. Jacobs, "The Caterer Problem," Naval Research Logistics Quarterly, I (June, 1954), pp. 154-65.

magnitude. Henderson and Schlaifer have listed applications as follows:[18]

1. Where to ship
2. Where to ship and where to produce
3. Where to ship, where to produce, and where to sell
4. Determination of the most profitable combination of price and volume
5. Determination of what products to make
6. Determination of what products to make, and what processes to use
7. Determination of method of obtaining lowest cost production

Harrison has indicated[19]

> The obvious type of application for linear programming is the construction of a numerical action schedule using available resources. In addition to this type of application, however, linear programming may be used, under certain circumstances, to aid in policy control. For example, a firm may have under consideration the investment of surplus capital to increase its material resources. Linear programming provides a technique for qualitatively estimating the theoretical increase in the selected measure of the firm's objective due to this proposed increase in material resources. As another example, the firm may wish to evaluate quantitatively the cost of some personnel policy, such as the elimination of scheduled overtime work. Linear Programming provides a technique for quantitatively estimating the cost of the policy change in terms of the firm's objective.
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> Quite aside from its use in determining numerical answers, linear programming can be a conceptual aid to the operations analyst. The linear programming approach, even if it is not carried through the data-gathering and computational phases, points up the necessity for clearly formulating objectives and focuses attention upon those parameters upon which optimization of the objective demands.

Charnes, Cooper, and Henderson state[20]

> In short, linear programming is concerned with the problem of planning a complex interdependent activities in the best possible (optimal) fashion. Problems of determining optimal product mix under given purchase and selling prices, machine productivities and capacities; problems of optimum storage, shipment, and distribution of goods; problems of minimizing set-up time in machine-shop operations; and problems of optimal labor allocation represent areas to which these techniques can be applied.

In the following chapter, some of the military applications of the foregoin principles are reviewed.

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[18]Henderson and Schlaifer, p. 76.

[19]Harrison, p. 237.

[20]A. Charnes, W. W. Cooper, and A. Henderson, An Introduction to Linear Programming, (New York: John Wiley & Sons, 1953), p. 1.

CHAPTER III

MILITARY USE

It is desired to make an efficient deployment of combat aircraft to competing activities or missions given a production schedule. The high priority aircraft are to go to the high priority missions. We want to maximize the number of aircraft deployed for combat, and simultaneously allocate enough aircraft to advanced flying schools to train crews to fly them. Minimization of surplus crews is another objective. Results are to be computed by mission and aircraft model first on a worldwide basis and then by area.[1]

Such an example as the above is obviously the sort of problem facing the military executive frequently. Admittedly the military situation can change rapidly in times of stress or actual war, but the sounder the initial plan to meet the problem the easier it is to modify it to suit new conditions and yet not disrupt both the combat and the training missions. Furthermore, once the method of solving the problem has been prepared and coded for use in an electronic calculator, solutions to meet new conditions can be obtained with but very little delay.

One of the most fundamental problems facing a top command planning staff - or indeed a staff at any level - is the question of logistic feasibility of a strategic plan. Limitations of availability and relative priority in competition with other plans of manpower, fuel, and ammunition, to cite but a few of the logistic factors concerned, may render the best strategic plan totally unfeasible. Indeed, a miscalculation of logistic factors may render

[1]Joseph V. Natrella, "New Applications of Linear Programming," a talk given before the National Meeting of the Association for Computing Machinery, 14-16 Sept., 1955, Philadelphia. Reproduced in pamphlet form by Directorate of Management Analysis, DCS/Comptroller, Hdqtrs., USAF.

carrying out of the plan disastrous; military history is most certainly replete with such cases. While at the present state of the science one cannot claim that mathematical programming will eliminate all the Commander's worries and do the work of his logistics staff, it is not incorrect to say that optimum feasible schedules are being obtained in a number of problems, albeit at this time modest ones.

The caterer's problem was mentioned[2] as being a paraphrased version of an aircraft spare engine problem. The military problem is this:[3]

> In order to accomplish its mission, a base requires monthly inputs of spare engines to meet specified engine change requirements. Engines removed for overhaul are transported from the base to an assigned repair depot. After they are overhauled, the engines are transported back to the base for possible reinstallment. If a monthly requirement for replacement engines is greater than the number of overhauled engines available at the base, new engines must be procured for that month. The base can utilize air or surface transport in moving engines to and from the depot. An engine which is airlifted to the depot is airlifted back to the base. Similarly, an engine surface transported is returned by surface lift. The total time an airlifted engine is in transit and repair is, of course, less than the time for a surface lifted engine. However, since airlift is an expensive means of movement, it must be used judiciously.
>
> The problem is to determine a schedule of monthly air and surface shipments of used engines that requires the minimum number of new engines to be procured and uses the maximum amount of surface lift. A shipping schedule that takes full advantage of fast transport - the one that uses airlift exclusively - will also require the minimum number of new engines to be procured. In general, there are other schedules that call for the same number of new engines, but use less airlift. Out of all these schedules, the one that uses the least airlift is the optimum solution to the problem.

A dynamic type of problem which arises in the military frequently is the determination of what rate of output to set for service schools to provide trained manpower for new weapons or tasks, or for some other type of future

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[2]Supra, p. 13.

[3]The Application of Linear Programming Techniques to Air Force Problems: A Non-technical Discussion, (Washington, D. C.: Directorate of Management Analysis, DCS/Comptroller, Hdqtrs., USAF, 17 December, 1954), p. 15.

requirement. This type is similar to the production and storage problem of the previous chapter. There is a need to find an optimum between (a) the extremes of steady output of trainees with waiting time in pools until in the early stages the equipment is ready and (b) the fluctuating output of trainees which just meets equipment schedules but disrupts efficient and economical training schedules.[4] Here again we have conflicting objectives for which the executive must decide the best middle course; mathematical programming can be a valuable aid to him in his decision.

Transportation type applications are many. Dantzig and Fulkerson have discussed the problem of meeting a fixed schedule of tankers in such a way as to minimize time in ballast.[5] Though the problem is a large one, it is mitigated by the fact that most variables are constrained to zero, that the minimizing form is particularly simple, and that even a large transportation type problem having no special features can be solved by hand using the Simplex algorithm, which method of computation is discussed in the next chapter. Flood has written on the same problem.[6] A complete set of voyages for tankers of the military fleet was computed so as to minimize the expected total distance to be traveled by the ships in ballast during the year ending 30 June, 1950, on deliveries according to the advance requirements estimates of the Armed Services Petroleum Purchasing Agency. Results gave 5% less ballast distance than a simple round trip schedule gave. Aside from the possible monetary savings, the critical factor of available tanker service is of considerable importance and interest to the transportation officer.

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[4] Ibid., p. 18

[5] G. B. Dantzig and D. R. Fulkerson, "Minimizing the Number of Tankers to Meet a Fixed Schedule," Naval Research Logistics Quarterly, I (September, 1954), pp. 217-22.

[6] M. M. Flood, "Application of Transportation Theory to Scheduling a Military Tanker Fleet," Journal of Operations Research Society of America, II (May, 1954), pp. 150-62.

More general articles on the shipping and transportation system problem have been written by Koopmans[7] and Heller[8], with the former discussing more the theoretical economic aspects of the problem than the application to a particular practical case. Koopman's article is a "readable" account of a monograph dealing in the mathematical theory.[9]

The requirement by the Congress that Armed Services purchasing be almost entirely on a bid basis and that contracts be awarded that are most advantageous to the government[10] has posed a knotty problem for many a contracting officer. A typical case best illustrates the truth of that statement. The Armed Services Textile and Apparel Procurement Agency (ASTAPA) often faces contract awards problems in which the possible combination of awards, with all the restrictions as to price and quantity, minimum and maximum lots of particular articles, et cetera, total tens of thousands. In one instance, 700 man hours were used to compute about a sixth of the possible ways of making awards; and it was estimated that 4000 man hours, or two man years of effort would have been required to compute all possible ways of making the awards.[11] Since a limited number of qualified personnel are available to make computations, it could conceivably take weeks or months to be sure that the awards are in fact most advantageous to the government - and it is questionable that the bidders would wait weeks or months. Indeed, common practice is of course for

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[7]Supra, p. 10.

[8]I. Heller, "Least Ballast Shipping Required to Meet a Specified Shipping Program," Symposium . . . ., pp. 164-71.

[9]George B. Dantzig, "Application of the Simplex Method to a Transportation Problem," Cowles Commission for Research in Economics, Monograph No.13, Activity Analysis of Production and Allocation, ed. T. C. Koopmans (New York: John Wiley and Sons, Inc., 1951), Chapter XXIII, pp. 359-73.

[10]U.S., Congress, Armed Services Procurement Act of 1947, Public Law 413, 80th. Congress, 1948.

[11]E. D. Stanley, et al., "Linear Programming in Bid Evaluation," Naval Research Logistics Quarterly, I(March, 1954), pp. 48-54.

a bidder to set a time limit, after which his bid is no longer valid at the prices quoted therein. Leon Goldstein has stated the problem thus:[12]

> Required to procure a given total quantity of each of a number of related items from producers. "Related" means that any producer can furnish any item but that his production of it is dependent on how much his facilities are strained by production of other items in the contract. Contractor may impose limitations:
>
> 1. Maximum and minimum dollar value of contract;
> 2. Maximum and minimum number of units of all items;
> 3. A per cent (less than 100%) of each item is acceptable, provided that the sum of all the percentages is less than 100%;
> 4. Capacity and price given on several items with a stipulation that only one item may be accepted.
>
> Bidders must also state cost per unit of each item and the maximum or minimum quantity he can deliver, or that he will deliver.

Although non-linear[13] conditions occur, in many cases they enter on a restricted scale and one can solve by computing the linear part over a selection of preassigned values and variables of the non-linear part. One point of interest brought out by Stanley, Honig, and Gainen is the following: if the optimum solution indicates that a bidder should receive less than his acceptable minimum award, the least cost to the government is the lesser of the costs of awarding him his minimum and of awarding him nothing.

The contract awards problem is an interesting one, and though of course not purely military, since one assumes that most businessmen want to minimize purchasing costs, is peculiar to the former because of the legal requirements existing to comply with the Congressional mandate. Mathematical programming probably offers the closest to a panacea here that it ever offers, and is coming into increasing use in this application. The very size and complexity of procurement problems in the government, and particularly the

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[12] Leon Goldstein, "The Problem of Contract Awards," Symposium . . ., pp. 147-154.

[13] I.e., price does not always stay constant for variation of quantity ordered. Non-linearity is further discussed in Chapter V.

military, all but demand mathematical programming be used for other than the simplest cases.

The Bureau of Ships, Department of the Navy, has applied mathematical programming to yet another problem, most closely related to the economics problem of too great a demand for too few resources. Electronics equipments must be allocated to fill all authorized allowance requirements aboard the ships and on the shore stations of the Naval Establishment. Equipments are seldom available in sufficient quantities to meet all requirements. In such cases, priority of ship types and projects is one of the controlling factors. There are other factors, usually conflicting and often seemingly unrelated which should be in some manner given consideration. For example, which ship should receive the one available set of a given type of equipment when

> A marginal substitute set is already installed on a high priority ship while a low priority ship has none installed to fill the requirement?
>
> A high priority project requires only a portion of the features built into the set while a low priority project requires and utilizes all of the set's features?
>
> The set may perfectly satisfy a low priority requirement but be only a temporary substitute for a high priority requirement?
>
> It is possible either to hold the set in storage for two years and then use it in its best possible application, or to install it immediately in a fairly good application?
>
> More than one set is required per ship and it is possible either to complete the allowance on a high priority ship or to give a low priority ships its first set?[14]

Until recently, solution to the above problems of allocation was done on a somewhat intuitive basis, with great consideration given to the arguments of the most persuasive talker. The problem seemed too complex to be susceptible to an orderly solution which would give all pertinent factors their proper

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[14] Daily Administrative Bulletin, Bureau of Ships, Department of the Navy, 3 February, 1955.

consideration. Experienced engineers in the Bureau were giving their time to solving such allocation problems when they might far better, and with considerably more enjoyment in so doing, have spent their time on technical problems to the considerable overall gain for the Fleet. Furthermore, use of new programming techniques will give an optimum use factor to the equipments available, and hence to the ships they go on, than has formerly been possible. The magnitude of the problem is shown by the fact that a single case may entail simultaneous consideration of 7 different kinds of projects, 1000 ships having 100 different kinds of sets already installed, and 10 kinds of related sets to be allocated in quantities from 1 to 300, phased in time over many months.

The Bureau of Ships has found it possible to assign weight factors to different users of the equipment, and show that the weights assigned are correct. The weights are assigned - though indirectly - by the operating forces, through answers to questions of what the forces desire in a series of specific cases. Analysis of the answers to the questions develops weight factors. It has even been possible to determine improvement factors to weigh the value of changing from one set to another in several cases, and to compare those several cases. The function maximized by mathematical programming is the overall use factor of the total equipments available, and the problem is almost exactly similar to the gasoline mixing problem of maximizing profit with a limited quantity of critical input items. The key to the use of programming technique was of course the determination of weight factors that would enable an evaluation of "payoff" to be made on a quantitative basis of concrete figures. Now that that problem has been solved, the technique can be extended to give other answers. The logical question that comes to mind next is to determine the best equipments to buy with a given amount of money

that has been made available through changed program requirements or for other reasons. The Bureau is currently working on this type of problem, the solution of which will give the Fleet its maximum return on the funds available for improvement of equipment.[15] And again, the indirect and unmeasurable return is that which will result from the fresing of highly trained technical personnel from what in the final analysis is merely a semi-clerical, non-technical chore, once the engineer has determined the applicable technical parameters. In these days of engineer shortages in every type of business in the country, the value of more efficient use of engineer time is of fundamental importance.

Finally, lest one think that the military not have its fifty-seven different varieties of problems in which programming by mathematical means can assist, mention is made of the counterpart of the Heinz problem of distribution. Certainly the mathematical symbols denoting shipping points and warehouses or destinations will not know the difference between a military warehouse and an industrial business warehouse, or whether the traffic manager is in or out of uniform. One can well imagine the value of minimizing the shipping cost of all aviation gasoline for the United States Air Force from the refineries where it is purchased to the depots and tank farms where it is to be stored against operational demands. Similarly, the shipping problem from depots holding identical equipment to a number of operational users at air or naval bases all over the country or the world can be tremendous in ton-miles of air lift or surface lift required.[16]

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[15]Interview with Commander R. H. Buck, USN, and Mr. Jack Smith, Electronics Division (Code 990), Bureau of Ships, Department of the Navy, December, 1955.

[16]<u>The Application of Linear Programming Techniques to Air Force Problems</u>.

CHAPTER IV

METHODS OF SOLUTION

> If business finds that it is important to solve problems of linear programming, it seems almost certain that means will be found of solving the great majority of the problems that occur.[1]

A number of methods of solution of mathematical programming problems have already been published, and the foregoing quotation is a vote of confidence in the mathematician's ability to evolve more methods as rapidly as needs arise. It would appear however that at the present time theoretical work is ahead of application, and more is known about mathematical programming than is generally put to use.[2] Presently published numerical methods include the following:

1. The Simplex Method[3]
2. The Method of Fictitious Play[4]
3. The Relaxation Method[5]
4. The Solution by a System of Ordinary Differential Equations[6]
5. A Numerical Method of J. von Neumann[7]
6. The Dual Method[8]
7. Transportation Procedure[9]

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[1]Henderson and Schlaifer, p. 100.

[2]Natrella, p. 2.

[3]Charnes, Cooper, and Henderson.

[4]G. W. Brown, "Iterative Solution of Games by Fictitious Play," Activity Analysis . . ., Chapter XXIV.

[5]Charnes, Cooper, and Henderson.

[6]G. W. Brown and J. von Neumann, "Solutions of Games by Differential Equations," Contributions to the Theory of Games, (Princeton, N.J.: Princeton University Press, 1950), ed. by Harry W. Kuhn and A. W. Tucker, pp. 73-9.

[7]J. von Neumann, "A Numerical Method to Determine Optimum Strategy," Naval Research Logistics Quarterly, I(June, 1954), pp. 109-15.

[8]C. E. Lemke, "The Dual Method of Solving the Linear Programming Problem," Naval Research Logistics Quarterly, I (March, 1954) pp. 36-47.

[9]Henderson and Schlaifer.

8. Profit-Preference Procedure[10]
9. Parametric Minimization Method[11]

The relative merits of various of the methods of solving problems have been given by several authors.[12, 13, 14]

As is so often true in various fields of applied science, the problems of greatest practical interest are the complicated problems; in the instance of mathematical programming, the cases involving a large number of variables and constraints are the practical problems. It is fortunate for this science that it has come about at the inception of development and use of electronic large scale digital computors, so that the high speed of computation serve to obtain solutions of problems in relatively short computational times. Of course, for the very reason that such computors are available, the techniques of mathematics evolved have emphasized simplicity and generality at the expense of a large amount of numerical manipulation.[15]

The best known method is the Simplex Method. This method requires as a starting point a feasible solution - by this is meant a mathematically feasible but not necessarily practical business solution. In the case of a mixing problem, for example, the initial solution might well be to use all of

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[10]A. Charnes, W. W. Cooper, D. Farr, et al., "Linear Programming and Profit-Preference Scheduling for a Manufacturing Firm," Journal of the Operations Research Society of America, I (May, 1953), pp. 114-29.

[11]S. I. Gass and T. L. Saaty, "The Parametric Objective Function," Journal of the Operations Research Society of America, II (August, 1954), pp. 316-19.

[12]A. Hoffman et al., "Computation Experience in Solving Linear Programs," Journal of the Society for Industrial and Applied Mathematics, I (September, 1953), pp. 17-33.

[13]A. Orden, "Solutions of Systems of Linear Inequalities on a Digital Computor," Proceedings of the Association for Computing Machinery, (Pittsburgh, May, 1952), pp. 91-5.

[14]A. J. Hoffman, "How to Solve a Linear Programming Problem," Proceedings of the Second Symposium in Linear Programming, II, (Washington, D. C.: DCS/Comptroller, Hdqtrs., USAF, with the National Bureau of Standards, January, 1951), pp. 397-424.

[15]Harrison, p. 235.

the restrained input items as excess, make none of the mixture, and hence have zero variable cost but zero profit. Fortunately for the decision-maker, however, the technique goes on from there. A sequence of feasible solutions is obtained, each one computed from its predecessor in a manner such that the best answer is gradually approached and that a previous solution is never repeated. For mathematical reasons that are beyond the scope of this paper to show, there are a finite number of solutions between the original one and the optimum one, and though one may not converge as rapidly as he might like to the optimum solution, he will converge. The operations are carried out by means of extremely simple, albeit repetitive and tedious arithmetic steps. The method is analogous to the elimination method of solving simultaneous equations, and as one can readily visualize, if there are fifty or more unknowns and equations, the process of solution is long drawn out. The method could be used for hand operations, or desk calculators, but of course for a problem of any size - and these problems have a way of becoming quite large very quickly - the time that would be required would be enormous, and resort to electronic computor aid must be sought.

Because the Simplex method requires considerable time to carry out, various shortcut methods have been devised. These shorter methods will however not work in all cases; but at least one mathematician is convinced that "one of the first tools to be involved in looking for a special method for a special problem is to see if the Simplex method, as applied to a particular problem, behaves in some striking way."[16] In one of the comparisons of methods previously referred to, that of A. Hoffman *et al*., the Simplex Method is indicated

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[16] Hoffman, p. 415.

as generally superior to the Method of Fictitious Play and the Relaxation Method. The final conclusions of the authors, however, are that any relative evaluation requires specification of the size of the problem, the accuracy demanded, and the amount of computation time reasonable to invest in obtaining this accuracy. For say a 7 x 7 mattrix, with accuracy to 4 or 5 decimals, the Simplex Method is outstanding.

A method proposed by Von Neumann, referred to simply as "A Numerical Method," has the advantage that as a part of the method one obtains in advance a guarantee as to both the maximum duration of the calculation and the maximum size of numbers that might occur in the course of the solution.

The dual of a problem was mentioned on page 10. Since solving the dual is to solve the problem, it is at times advantageous to apply solution techniques to the dual rather than to the problem itself. It may be that fewer equations result from consideration of the dual, or that other mathematical relationships result which reduce the computational time of the dual with respect to the primal, or original problem. It is obvious from the foregoing that this method of attack is only applicable in certain special cases.

The most frequently useful of the abbreviated procedures is the Transportation Problem Procedure which obtained its somewhat misleading name because of the original application to determine a lowest cost shipping program. It can be used in many other fields not involving transportation, but which are reduced by mathematics to the same type of equations. And of course there are some transportation problems which the method cannot solve in spite of its name. The method is extremely simple, and as one becomes more familiar with it, he can use further shortcuts to reduce more complex problems to a form that can be similarly handled. Appendix A gives detailed instructions for its use, and carries

to a solution, through all steps, the problem of Chapter I concerning the empty freight cars.

The Profit-Preference Procedure of Charnes, Cooper, Farr, et al., though restricted in the class of problems it can treat, is simple to use. The scheduling done by SKF, referred to previously, uses this method. But as one might expect, the simplification achieved is at the expense of loss of some information which would be obtained by the use of a more general method. Lost information includes alternate optima, second best programs, etc., The method consists essentially of forming a table of machine preferences for making each of the products possible to make on the given machine, the preference being expressed in terms of profit. Allocation of machine times is then determined on the basis of maximum overall profit, with final checks being made to ensure that all restrictions (such as required production to meet sales orders) have been met; if they have not, adjustments are made, again by using profit-preferences. The method is more fully explained in Appendix B. The method is impractical if the structure of the problem becomes overly complicated, and resort to the more general procedure must then be had. But when it can be used, the procedure is fast and flexible as well as simple.

The Parametric Minimization Method given by Gass and Saaty is a procedure for furnishing the snswers to a particular type of problem which is best described by an example taken from the military. Suppose that in a shipping problem a weight factor must be applied to theaters of operation, such as most certainly was the case in World War II as regards the conflicting demands of the European and the Far Eastern Theaters. The mathematician cannot of course assign the weights; but he should be able to show the command which does so the effect of various weight assignments on the answers. He can do so by Gass and Saaty's procedure. This problem is of course similar to the

one facing the Bureau of Ships in the allocation of Electronic equipments, with weight factors being given to various different uses for a type of equipment.

Jacob's discussion of the caterer problem points out that it is sometimes possible to find a transformation which converts the linear programming problem into a related but simpler problem with the same optimum solution, although the related problem may have feasible solutions for which no feasible solutions correspond in the original problem. He hopes and expects that a criterion for reduction can be found in terms of the coefficients of the constraints and in the signs of the terms; if the criterion permits recognition of cases where problems too large to handle by general procedures can be brought into range of computors, the applications of linear programming to dynamic systems will be materially extended.

It appears that methods other than some version of the Simplex Method such as fictitious play, or relaxation, converge too slowly to be of great use unless, quite conceivably, one finds imprecise answers satisfactory. Of course, future experience in particular programming problems may lead to a somewhat different conclusion; and at the present time there are types of problems for which Simplex Techniques are unsuitable, but for which iterative or repeated step methods are quite suitable. The type of computor available may also dictate the method employed. By-and-large, however, experience to date has indicated that the Simplex Method should most probably be used.

CHAPTER V

LIMITATIONS

Limitations of the technique of applying mathematical programming are twofold in nature; mathematical and practical. The practical difficulties are those commonly associated with applying any pure science to business problems. One finds it difficult to express in suitable measurable terms the objectives and controlling factors, or constraints; mathematical programming has no monopoly on this difficulty, and it is of course a difficulty associated with the particular operation under investigation and study, not with the technique being employed to investigate it. Secondly, even when one is able to find suitable and measurable terms for the operation, it may well be exceedingly difficult to determine proper numerical values for the coefficients of the terms. Like any other problem, if a more precise answer is needed, more precise data must be used, and that in turn entails a thorough and detailed study of the problem. Finally, there is the computational labor required to solve the problem, once it has been set up in measurable terms with proper coefficients. The very fact that mathematical programming techniques are being used to investigate a given operation is indication in itself that the problem is quite complex, with many variables and constraints, such that it does not lend itself to quick solution by other means, or by inspection. We must necessarily expect them in cases of application of the technique that computational labor may be immense, even for electronic calculators; must accept that fact as the nature of the beast; and be thankful that there <u>is</u> a technique to help us where until recently there was nothing but educated

guessing. "Thus the practical difficulties encountered in applying linear programming are in a sense an indication of the level of difficulty of the problems which linear programming can treat."[1]

The mathematical difficulties are discussed in some detail by Dorfman.[2] He points out that the limitations are intrinsic, and based on the postulates of linear programming. These postulates have been called (1) linearity, (2) divisibility, (3) additivity, and (4) finiteness, and are peculiar to the technique itself. They are defined as follows by Dorfman:

> 1. Linearity. By definition, in linear programming each process is characterized by certain ratios of the quantities of the inputs to each other and to the quantities of each of the outputs. These ratios are defined to be constant, and independent of the extent to which the process is used. Stated somewhat oversimply, twice the output of a product costs twice as much in variable costs and uses twice as much of each of the inputs making up the product.
>
> 2. Divisibility. It is assumed that any process can be used to any positive extent so long as sufficient resources are available; indivisibilities and "lumpiness" in production are ignored.
>
> 3. Additivity. It is assumed that two or more processes can be used simultaneously, within the limitations of available resources, and that if this is done the quantities of the outputs and inputs will be the sums of the quantities which would result if the several processes were used individually.
>
> 4. Finiteness. It is assumed that the number of processes is finite.

Although on first examination, the limitation of linearity appears unduly restrictive, in practical application such is not the case. In recent years, there have been developed techniques for dealing with non-linear processes to a limited scale. Thrall states:

The many successful applications of linear programming to problems

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[1] Harrison, p. 235.

[2] R. Dorfman, Application of Linear Programming to the Theory of the Firm, (Berkeley, Calif.: University of California Press, 1951), pp. 80-1.

of scheduling, allocation, inventory control, and related topics have stimulated research on effective techniques for handling linear systems. Although both the theory and the application of linear programming still contain many promising areas for study it now seems appropriate to increase the amount of attention given to non-linear programming. Indeed, many of the problems currently handled by linear programming are linear only to a first approximation and many other important problems do not even have reasonable linear approximations.

In general one can divide non-linear programming problems into two classes: (1) problems in which the central interest is in establishing the existence of a solution and (2) problems in which the existence of solutions has been established but effective means of calculating them are needed.

In this paper my concern is with the second of these classes.[3]

He goes on to examine the "parimutual problem" of Mr. R. Isaacs[4] in which nonlinearity occurs since the bettor by the size of his bet changes the odds on his selection. In case one gets carried away with the thought that here is a way to beat the horses, I hasten to add one additional comment from Thrall: "In the limit the expected gain of each of the 'last' bettors is zero; all of the potential gain has then been absorbed into the take of the track."[5]

The postulate of divisibility is not seriously restrictive unless in some instances, few in number, such as shipbuilding the product of the enterprise consists of a very few items. In these instances there is small probability that mathematical programming is directly applicable anyway. Nor is additivity greatly restrictive, though apparently violated at times. In the integrated steel company, for example, the processes of pouring ingots and rolling blooms give different outputs if performed separately, because of the considerable reheating required, than if performed in immediate sequence so that but very little reheating or soaking is required. Dorfman suggests that this

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[3]R. M. Thrall, "Some Results in Non-Linear Programming," Proceedings of the Second Symposium . . ., p. 471.

[4]R. Isaacs, "Optimal Horse Race Bets," American Mathematical Monthly, LX (1953), pp. 310-15.

[5]Thrall, p. 493.

case is in fact a new system, not several parts of an old system, and that after all, additivity is an assertion of possibility, not a guarantee of efficiency.

Probably the most limiting of the four postulates enumerated is that of finiteness. In agriculture, to name but one of many industries, the range of alternatives is essentially infinite. The chemical and oil refining industries are similarly hardly limited in their alternative possibilities. It is possible normally however to approximate a particular situation to almost any desired degree of accuracy and refinement by use of a finite, albeit large number of separate and discrete processes.

Certain assignment problems lead one to another limitation of the programming technique; that of scheduling. In the choice of machine tools, for example, one may develop by mathematical programming the optimum manner to use the various tools available, but not the order in which to use them. It seems probable that further attention on this phase of the assignment problem will be forthcoming, and that techniques will be developed to reduce the scheduling of the optimum choice of tool to a routine procedure.

A type of limitation that should be mentioned is that occurring because of the integer nature of the solutions required. An excellent example is clearly illustrated and followed through in detail in J. M. Danskin's article on an example of/failure of linear programming.[6] The problem is one of determining the proper types of ammunition to carry in an aircraft carrier's magazines to maximize her effectiveness against a variety of tasks, under the assumption that magazine replenishment between tasks is not possible. The limitation arises from the necessary condition that one cannot divide a 2000 lb. bomb into halves

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[6]J. M. Danskin, "Linear Programming in the Case of Uncertainty: Example of a Failure," Proceedings of the Second Symposium . . ., pp. 39-53.

or quarters, either for loading the bomb racks of the planes or for stowing the magazines. A second difficulty, which is of the type referred to previously as the practical difficulty of assigning suitable measurable terms to the objective, occurs in the rather vague nature of the return on the investment, or the payoff of the bomb load. Danskin draws the conclusion that we are warned that mathematical programming cannot solve all our problems, powerful tool though it may be; thus the decision-maker is often thrown back on applying his judgment and experience without the aid of quantitative problem analysis. The foregoing reservation seems an apt note on which to conclude.

APPENDIX A

Directions for Solving Problems by the Transportation Problem Procedure

This example consists of assigning the manner of shipping railroad empties from three shipping points having excess number of empties to five destinations having, in total, the same number of freight car shortages. The data for the problem is given in Table A below; that is, the shipping rates from each shipping point to each destination, the excesses at each shipping point, and the shortages at each destination.

(A)	D1	D2	D3	D4	D5	Surplus Totals
S1	$10	20	5	9	10	9
S2	2	10	8	30	6	4
S3	1	20	7	10	4	8
Shortage Totals	3	5	4	6	3	21

It is first necessary to get a feasible solution, that is, one which meets the fixed requirements (of excesses and shortages) regardless of cost. This solution may be obtained as follows. Take the 9 cars at shipping point S1 and fill the shortages of 3 cars at D1 and 5 at D2; put the remaining car at D3. Take the 4 cars at S2 and put 3 more at D3 to complete the shortage there of 4, and put the fourth car at D4. Take the 8 cars at S3, put 5 at D4 to complete the shortage there of 6 and put the remaining 3 at D5, which fills the shortage there of 3. The procedure could obviously be used to fill a table of any size. This solution is entirely feasible and would cost $351.
(3 x $10 + 5 x $20 + 1 x $5 + 3 x $8 + 1 x $30 + 5 x $10 + 3 x $4 = $351)
The table would then be shown in Table B.

(B)	D1	D2	D3	D4	D5	
S1	3	5	1			9
S2			3	1		4
S3				5	3	8
	3	5	4	6	3	21

Since we learned in Chapter 4 that the method we are using leads to the optimum solution by proceeding through a series of solutions, each closer to the optimum than its predecessor, and never repeating, it is obvious that we may save ourselves a number of computational steps if we obtain a first solution which is the most economical we can get by quick inspection. Stated differently, if we use common sense to get a starting point which is fairly inexpensive in terms of shipping costs, we will save ourselves a great deal of

computational effort, even in a problem as short and simple as the example we are using for instruction purposes. Therefore let us attack the problem as follows. Take any shipping point at all and use the capacity (number of empties) there to fill those destination requirements (shortages) which by the cost of shipping it seems most economical to fill. When that capacity is exhausted, take another shipping point and do the same, first filling any requirements that were only partially satisfied by the preceding shipping point. Using this procedure we can proceed as follows. Take the 8 cars at S3 and put 3 at D1 at $1 each; put 3 of the remaining at D3 at $4 each; and the other 2 at D3 at $7 each. Fill D3's remaining requirements with 2 cars from S2, and put the remaining 2 from S2 at D2 (both D1 and D5 cost less but are already full from the shipment from S3). Finally, complete the table by placing the 9 cars from S1, 3 at D2 and 6 at D4. We then have Table C, at a cost of $179, which is considerably better than our random solution of Table B, and therefore offers us fewer steps to go through to achieve the optimum solution.

(C)	D1	D2	D3	D4	D5	
S1		3		6		9
S2		2	2			4
S3	3		2		3	8
	3	5	4	6	3	

It is next necessary to build up a type of cost table. First, fill in a table showing the actual shipping costs, taken from Table A, for those shipments which are actually in use in Table C. This gives Table D. We can refer to the "squares" of the table as "square S1D1" etc., and in Table D, S1D4 is equal to $9, or S1D4 = 9.

(D)	D1	D2	D3	D4	D5
S1		20		9	
S2		10	8		
S3	1		7		4

Second, in building up the cost table, of which Table D is the first step, assign row and column values on an arbitrary basis; row and column values are the values shown under those headings in Table E. They are obtained by assigning a value to row S1, such as zero, or 1, or 2, or anything (we have chosen zero) and then under every square of row S1 which contains a rate (S1D2, S1D4, etc.) assign a column value such that the sum of the row and the column values equals the sum of the value in the table. Thus, we obtain Table E, in which S1D2 is 20, and since the row value was zero, the column value must be 20; S1D4 is 9 and therefore the column value under D4 must be 9 so that the row value zero and the column value 9 equal the square S1D4. It should be noted that both positive and negative values of row and column values may occur. In observing S2D2 we see that we already have a column value of 20, and that the square value is 10, and therefore the row value for S2 must be -10. We then proceed to determine the column value for D3, since we have a row value of -10 and a square value of 8 for S2D3; the column value is necessarily 18. Table E is completed by similar steps to give the values shown.

(E)	D1	D2	D3	D4	D5	Row Values
S1		20		9		0
S2		10	8			-10
S3	1		7		4	-11
Column Values	12	20	18	9	15	

Finally, we make Table E into a cost table, Table F, by filling in all the blank squares with the sum of the developed row and column values as they are shown in Table E. S1D1 then becomes 12, the sum of the S1 value of zero and the D1 value of 12, etc.

(F)	D1	D2	D3	D4	D5	Row Values
S1	12	20	18	9	15	0
S2	2	10	8	-1	5	-10
S3	1	9	7	-2	4	-11
Column Values	12	20	18	9	15	

As one developes familiarity with the method, he omits the majority of the tables which have been added for instructional purposes. All that is needed is a rate table, Table A; a route table, Table C; and a cost table, Table F, for the first stage of the method.

It is now possible to determine what change in our route of Table C should be made to reduce our shipping cost. Examine the table in comparison to the rate table, and find the square where the figure in Table F is larger by the greatest amount than the figure in Table A. By inspection this is square S1D3 in which we find a value of 18 compared to Table A's value of 5. We now know that a change in the route so that more cars go from S1 to D3 we will save $13 on each additional car that we can put on this route. A detailed explanation of the reason for this statement may be found in Henderson and Schlaifer's article. The problem is now to find out how many cars we can shift to square D1D3, i.e., how many cars we can send from shipping point S1 to destination D3 instead of using the solution of Table C. We do so as follows. Construct Table G, and in square S1D3 write +x: this is the unknown amount which will be shipped from S1 to D3. The other amounts in Table G are obtained by copying Table C.

(G)	D1	D2	D3	D4	D5	
S1		3	+x	6*		9
S2		2	2			4
S3	3*		2		3*	8
	3	5	4	6	3	

But we must know which routes are unaffected if we are to complete the table properly. We mark those squares in Table G, after copying the Table C values, with an * or some mark, which is the <u>only</u> number in either its row or its column. The +x in S1D3 is counted as a number for this purpose. Hence we

mark the value S3D1 with an *, since 3 is the only number in the row D1; and we mark the 6 of S1D4 and the 3 of S3D5 with an *. We next consider the * numbers as not in the table, and go through the * procedure again, looking for any numbers which are now alone in either rows or columns. Thus we * square S3D3, where the 2 is the only number meeting the requirement of being alone, in this case in row S3. We cannot * any more numbers. We now have Table H, except that the only square with an x value in it is the +x in S1D3. We complete Table H by disregarding all the * values and filling in +x and -x values such that the rows and columns add up properly to the totals shown, S1D2 must necessarily become 3-x so that the sum of the three values of row S1 (3-x, +x, 6) add up to the known total 9.

(H)	D1	D2	D3	D4	D5	
S1		3-x	+x	6*		9
S2		2+x	2-x			4
S3	3*		2*		3*	8
	3	5	4	6	3	

The maximum number of cars we can divert to S1D3 is then 2, for if we diverted any more we should have a minus number of cars for square S2D3, where our new value is 2-x; one of the postulates of our problem is of course that we do not want negative shipments, i.e., shipments from destinations to shipping points. This restraint is normally referred to as a "non-negativity" restraint. By substituting the value of 2 for x, we obtain Table J (Table I omitted) and can determine that the shipping cost is now only $154, or an improvement over our value of $179; and as we said, we save 2 cars at $13 each, or $26 over the first solution, or $179 - $26 = $154.

(J)	D1	D2	D3	D4	D5	
S1		1	2	6		9
S2		4				4
S3	3		2		3	8
	3	5	4	6	3	

It is now necessary to repeat the same procedure, beginning with Table J, as we have just done in beginning with Table C. Tables K and L are intermediate steps to obtaining Table M. In Table K, a row value of 10 for row S1 was taken, and the complete table developed just as was done for Table F. Comparison of Table K with Table A shows that S3D2 in the former is higher in value by 2, and therefore that we will save $2 for each car that we can ship over that route if we make the proper adjustments in the other routes. Table L, developed just as was Table H, shows that we can ship 1 car over route S3D2. Our new routing table is then Table M, and its cost is $151, or $2 less than our previous routing Table J.

(K)	D1	D2	D3	D4	D5	
S1	-1	20	5	9	2	10
S2	-11	10	-5	-1	-8	0
S3	1	22+2	7	11+1	4	12
	-11	10	-5	-1	-8	

(L)	D1	D2	D3	D4	D5		
S1		1-x	2+x	6*		9	
S2		4*				4	
S3	3*	+x	2-x		3*	8	x =1
	3	5	4	6	3		

(M)	D1	D2	D3	D4	D5	
S1			3	6		9
S2		4				4
S3	3	1	1		3	8
	3	5	4	6	3	

Once more we go through the procedure, obtaining Tables N and O in reaching Table P, and improve our cost to $150.

(N)	D1	D2	D3	D4	D5	
S1	-1	18	5	9	2	9
S2	-9	10	-3	1	-6	1
S3	1	20	7	11+1	4	11
	-10	9	-4	0	-7	

(O)	D1	D2	D3	D4	D5	
S1			3+x	6-x		
S2		4*				x=1
S3	3*	1*	1-x	+x	3*	

(P)	D1	D2	D3	D4	D5
S1			4	5	
S2		4			
S3	3	1		1	3

Another trial, giving Table Q, shows that in no case is there any value in Table Q which is higher than its corresponding square in Table A. We know then that we have, in Table P, reached the optimum arrangement. There remain no changes we can make in that table to obtain a lower cost of making the shipments of empty freight cars that Table A gave as being required. We have given the freight car dispatcher the information on which to make his decision; should he decide, for reasons best known to him, to make a shipment other than Table P shows, he can compute the excess cost to do so, and base his decision on concrete cost data, not conjecture.

(Q)	D1	D2	D3	D4	D5	
S1	0	19	5	9	3	5
S2	-9	10	-4	0	-6	-4
S3	-1	20	6	10	-4	6
	-5	14	0	4	-2	

APPENDIX B

Directions for Solving Problems by the Profit-Preference Procedure

This example, taken from the Charnes et al. article referred to on page 28 of the text, examines a scheduling problem in which 6 products are to be manufactured using, among other things, 8 screw machines and 3 grinders, these being the bottleneck items in the manufacturing processes. This reduction to 11 machines was a part of the simplification process used by the linear programmers to reduce the problem to a workable size; reference to the Charnes et al. article should be made by the reader desiring to go into the fine points of the procedure. The reduction to but 6 products was a similar device; actually, the 6 represented 6 groups of products rather than merely 6 products. Determination was then made of the profit to be had from production of each unit of each product; of the time each product would require if made by various processes (using various of the screw machines, one per process, on which the job could be done; plus various of the grinding machines, also one at a time, if and when grinding was needed for finishing operations); of the capacities of the 11 machines; of the units of each product required to meet sales demands. These data are shown in Table I, pages 44-45.

It is next necessary to construct a simple profit-preference table; and this is shown in Table II, which results from application of the following formula:

$$\left.\begin{array}{l}\text{Total profit capacity of any S}\\ \text{or G used exclusively on any P}\end{array}\right) = \frac{\text{rated available hours}}{\text{estimated unit time}} \times \text{unit profit}$$

For example, reference to Table I shows that total profit capacity for S1, used exclusively on P5, is given as (942/ .17127) x 21.1429 =\$116,280, the figure displayed in the first cell of Table II, contrasted with (942/.00223) x .2125= \$89,760 for S1 used on P1. Obviously, S1 has a preference, on a profit basis, for P5 over P1; and computation of its profit in making P6 shows that it also prefers P5 over P6, the latter figure for profit being \$102,680. Hence Table II shows a 1, 2, 3 ranking preference of S1 for P5, P6, and P1 in that order, and with the values shown. Other cells are similarly determined.

Program calculation using Table II are then obtained as follows: (1) Column 1 for screw machines alone is summed to yield total profits if grinders are no restriction on screw machine capacities: approximately \$1,170,190. (2) Column 1 for grinding machines alone is similarly added to obtain total profit from product going through these grinders on the assumption that screw machines can adequately take care of their output: approximately \$341,870. (3) The products going through screw machines which also require grinders are totalled to yield a profit capacity of \$549,800 compared to \$341,870 of pure grinder profit potential. It is clear that profit-wise the grinders restrict the screw machines rather than the reverse. (4) To get maximum plant profit potential the products

from grinders first require distribution among screw machines in order to utilize the more restricted profit bottleneck to its maximum value.

Note that P5 is optimal on G1 and G3 and also that P6 is the only one which can be made on G2. Therefore, different screw machine types should be assigned so that production of one of these products by a screw machine does not inhibit production of the other.

Since P5 is optimal on both S1 and S8, and the grinders will take their total production, $116,280 + $119,680 < $119,340 + $119,680, P5 is allocated to these two screw machines. The remainder of G1 capacity can then be used for P6 from, necessarily, S5, S6, or S7.

G2 can take $102,850 of P6. Both S6 and S7 have better alternative uses than S5. Hence S5 is used to full capacity for P6. The remaining capacity of G2 and G1 for P6 is then drawn from S7 since S6 has a better alternative use than S7. The full capacity of S6 can then be employed for P2: $75,225, a better result than would have occurred if either S5 or S7 had to be so allocated to their next best use.

The remaining capacity of S7 can then be employed on its next best use, P3 = $39,800, since 97.4% of its capacity after allocation of P6 is available for this production. The remaining screw machines S2, S3, and S4 are then programmed for P1, P1, and P3, on the assumption carried throughout this analysis that adequate grinder capacity is available to handle this output.

The optimal plant profit potential program so calculated is given in Table III: as a result of the grinder requirements the optimum program is reduced by nearly $100,000 per month from $1,170,190 for screw machines alone to $1,076,645 for screw machines and grinders together.

It is next necessary to see whether sales restrictions, or sales demands have been met. Comparison between Tables I and III shows that demands are not met in the case of P4, where 3,500 units are required and under the optimal program of Table III none are scheduled. It is obvious that since only G1 can produce P4, G1 must be used, and almost all of its capacity at that, since G1 can make only 105/.0292 = 3590 units, and 3500 are needed, or 97.6% capacity. As a matter of fact, a more careful examination of our data would have shown before we developed Table III that we must necessarily use almost all of G1 to produce the required 3,500 units of P4, whether it was the most profitable thing to do or not. But by developing Table III we have determined an optimal plant profit potential program, and can subsequently measure the cost to us of having to meet the various sales minima given us.

This reallocation of G1 to P4 requires further adjustment of all other products which utilize the limited capacity of the other grinder types. The program of Table IV is finally obtained, with an indicated profit of $1,039,600. It is obvious that the addition of the P4 sales requirement caused a lost profit of $37,045 additional to that already lost by other minimum requirements.

Comparison of the figures in this appendix will show minor differences in Tables III and IV from those of the referenced article; errors have crept

into the Charnes et al. tables, and the tables of this appendix contain new figures believed to be correct.

As was mentioned in the text, this profit-preference procedure will prove impractical if the problem structure becomes sufficiently complicated. But then recourse may be had to other techniques such as the Simplex Method. It is interesting to note that Charnes et al. state that Simplex calculations for this problem revealed the presence of alternate optima. But for simple problem structure relatively simple calculations, capable of being made by clerical personnel, once the cost and time parameters have been determined, offer a quick and positive solution to this type of scheduling problem. As has been mentioned previously, SKF uses such a method and schedules on a weekly basis, allowing each time for new work and uncompleted orders still in process.

TABLE I

Product Profits, Manufacturing Times, Machine Capacities, and Sales Requirements for Sample Problem

Machine	Cap'y, Hrs/Mo	Product										
		P1				P2				P3		
		Process Times, hours				Process Times, hours				Process Times, hours		
		1	2	3	4	5	6	7	8	9	10	11
S1	942	.00223										
S2	1050		.00249			.00897						
S3	2280			.00525			.01892					
S4	2127				.00504			.01818		.00824		
S5	68											
S6	978								.00811		.03938	
S7	553											.02286
S8	1115											
G1	105											
G2	12											
G3	702											

		P4				P5				P6			
		12	13	14	15	16	17	18	19	20	21	22	23
S1	942					.17127	.17127			.10467	.10467	.10467	
S2	1050												
S3	2280												
S4	2127												
S5	68	.01943											.00756
S6	978		.27117										
S7	553			.15346									
S8	1115				.30926			.19680	.19680				
G1	105	.02920	.02920	.02920	.02920	.01858		.01858		.01136			.01136
G2	12										.00133		
G3	702						.12398		.12398			.07577	

Machine	Cap'y, Hrs/Mo	Product										
		P6										
		Process Times, hours										
		24	25	26	27	28	29	30	31	32	33	34
S1	942											
S2	1050											
S3	2280											
S4	2127											
S5	68	.00756	.00756									
S6	978			.10546	.10546	.10546						
S7	553						.05968	.05968	.05968			
S8	1115									.12027	.12027	.12027
G1	105			.01136			.01136			.01136		
G2	12	.00133			.00133			.00133			.00133	
G3	702		.07577			.07577			.07577			.07577

Product	Profits/Unit, $	Sales Requirements, Units/Month
P1	0.2125	422,000
P2	0.6239	117,000
P3	1.6932	200,000
P4	8.7652	3,500
P5	21.1429	5,500
P6	11.4053	9,000

TABLE II

Profit Preference Schedule
($)

Machines	Profit Rank				
	1	2	3	4	5
S1	P5 116,280	P6 102,680	P1 89,760		
S2	P1 89,590	P2 73,440			
S3	P1 92,140	P2 75,140			
S4	P3 438,600	P1 89,590	P2 73,440		
S5	P6 102,590	P4 30,685			
S6	P6 105,740	P2 75,225	P3 41,990	P4 31,620	
S7	P6 105,570	P3 41,140	P4 31,535		
S8	P5 119,680	P6 105,740	P4 31,620		
G1	P5 119,340	P6 105,400	P4 31,450		
G2	P6 102,850				
G3	P5 119,680	P6 105,740			

TABLE III

Optimal Plant Profit Potential Program

Product	Grinders	% Cap.	Screw Mach.	% Cap.	Units of Product		$ Profit
P1			S2	100	421,500)	855,000	89,590
P1			S3	100	433,500)		92,140
P2			S6	100	120,500		75,225
P3			S4	100	259,000)	282,500	438,600
P3			S7	97.4	23,500[g])		39,800[g]
P4					0		0
P5	G1	97.4[c]	S1	100	5,500[a])	11,160	116,280
P5	G3	100	S8	100	5,660[b])		119,680
P6	G2	99.8[d]	S5	100	9,000[d])	9,240	102,590
P6	G2	negl.	S7	-	-)		-
P6	G1	2.6	S7	2.6[f]	240[e])		2,740[e]
Total Profit							$1,076,645

Sample Calculations. Data from Tables I and II.

a) 942/.17127 = 5,500

b) 1,115/.19680 = 5,660

c) 5,500 x .01858/105 = 97.4%

d) 68/.00756 = 9,000; 9,000 x .00133/12 = 99.8%; neglible time remains for use of G2 on other products.

e) 2.6% x 105/.01136 = 240; 240 x 11.4053 = $2,740

f) 240 x .05968/553 = 2.6%

g) 97.4% x 533/.02286 = 23,500; 23,500 x 1.6931 = $39,800

TABLE IV

Optimal Program Meeting All Sales Requirements

Product	Grinders	% Cap.	Screw Mach.	% Cap.	Units of Product		$ Profit
P1			S2	100	421,500)		89,590
P1			S3	100	433,500)	1,260,000	92,140
P1			S1	96.0	405,000)		86,200
P2			S6	100	120,500		75,225
P3			S4	100	259,000)	259,630	438,600
P3			S7	2.6	630)		
P4	G1	97.6	S5	100	3,500		30,685
P5	G3	100	S8	100	5,660)	5,880	119,680
P5	G1	2.4	S1	4.0	220)		4,650
P6	G2	100	S7	97.4	9,020		102,850
Total Profit							$ 1,039,600

BIBLIOGRAPHY

Books

Charnes, A., Cooper, W. W., and Henderson, Alexander. An Introduction to Linear Programming. New York: John Wiley and Sons, Inc., 1953.

Dorfman, Robert. Application of Linear Programming to the Theory of the Firm. Berkeley, California: University of California Press, 1951.

Koopman, Tjalling C. (ed.). Activity Analysis of Production and Allocation. Cowles Commission for Research in Economics, Monograph No. 13. New York; John Wiley and Sons, Inc., 1951.

Kuhn, Harry W., and Tucker, A. W. (ed.). Contributions to the Theory of Games. Princeton, New Jersey: Princeton University Press, 1950.

McCloskey, Joseph F., and Trefethen, Florence N. (ed.). Operations Research for Management. Baltimore: The Johns Hopkins Press, 1954.

Morse, Phillip M., and Kimball, George E. Methods of Operations Research. Cambridge, Massachusetts: Technology Press of the Massachusetts Institute of Technology and John Wiley and Sons, Inc., New York, jointly, 1951.

Owens, Richard N. Introduction to Business Policy. Homewood, Illinois: Richard D. Irvin, Inc., 1954.

Von Neumann, John, and Morgenstern, Oscar. Theory of Games and Economic Behavior. Princeton, New Jersey: Princeton University Press, 1944.

Williams, John D. The Compleat Strategyst. New York: McGraw-Hill Book Company, Inc., 1954.

Public Documents

U. S. Congress. Armed Services Procurement Act of 1947. Public Law 413, 80th. Congress, 1948.

Reports

Magee, John F. Studies in Operations Research. I: Linear Programming in Production Scheduling. Prepared under the auspices of Arthur D. Little, Inc., Cambridge, Massachusetts, n.d. (Mimeographed.)

Office of the Comptroller, Headquarter, U.S. Air Force. Project SCOOP, Scientific Computation of Optimum Programs, Symposium on Linear Inequalities and Programming. Washington, D.C., 1952. In conjunction with the National Bureau of Standards.

________. Proceedings of the Second Symposium in Linear Programming. Washington, D.C., 1955. In conjunction with the National Bureau of Standards.

Articles and Periodicals

Charnes, A., Cooper, W.W., and Mellon, B. "Blending Aviation Gasolines," Econometrica, XX (April, 1952), 135-59.

Charnes, A., Cooper, W.W., Farr, D., et al. "Linear Programming and Profit-Preference Scheduling for a Manufacturing Firm," Journal of the Operations Research Society of America. I(May, 1953), 114-29.

Christensen, R.P., and Mighell, R.L. "Food Production Strategy and the Protein-Feed Balance," Journal of Farm Economics. XXXIII(August, 1951), 183-91.

Dannerstadt, Gunnar. "Production Scheduling for an Arbitrary Number of Periods Given the Sales Forecast in the Form of a Probability Distribution," Journal of the Operations Research Society of America, III(August,1955), 300-318.

Dantzig, George B., and Fulkerson, D.R. "Minimizing the Number of Tankers to Meet a Fixed Schedule," Naval Research Logistics Quarterly, I(September, 1954), 217-22.

Dorfman, Robert. "Mathematical, or 'Linear' Programming," American Economic Review, XLIII(December, 1953),797 ff.

Factory Management and Maintenance. CXII(January, 1954), 136-7. "New Machine Loading Methods."

Flood, M.M. "Application of Transportation Theory to Scheduling a Military Tanker Fleet," Journal of the Operations Research Society of America, II(May,1954), 150-62.

Gass, S.I., and Saaty, T.L. "The Parametric Objective Function," Journal of the Operations Society of America, II(August, 1954), 316-19.

Henderson, Alexander, and Schlaifer, Robert. "Mathematical Programming," Harvard Business Review, XXXII(May-June, 1954), 73-100.

Hitchcock, F.L. "The Distribution of a Product from Several Sources to Numerous Localities," Journal of Mathematics and Physics, XX(1941), 224-30.

Hoffman, Alan, et al. "Computation Experience in Solving Linear Programs," Journal of the Society for Industrial and Applied Mathematics, I(September, 1953), 17-33.

Isaacs, R. "Optimal Horse Race Bets," American Mathematical Monthly, LX(1953), 310-15.

Jacobs, W.W. "The Caterer Problem," Naval Research Logistics Quarterly, I(June, 1954), 154-65.

Koopmans, Tjalling C. "Optimum Utilization of the Transportation System," Econometrica, XVII Supplement (July,1949), 136-45.

Lemke, C.E. "The Dual Method of Solving the Linear Programming Problem," Naval Research Logistics Quarterly, I(March, 1954), 36-47.

Orden, A. "Solutions of Systems of Linear Inequalities on a Digital Computor," Proceedings of the Association for Computing Machinery, (May, 1952), 91-5.

Railway Age. CXXXIV(April 20, 1953), 73-4. "Operational Research in Distributing Empty Cars."

Samuelson, P.A. "Spatial Price Equilibrium and Linear Programming," The American Economic Review, XLII, 3.

Stanley, E.D., et al. "Linear Programming in Bid Evaluation," Naval Research Logistics Quarterly, I(March, 1954), 48-54.

Stigler, George F. "The Minimum Cost of Subsistence," Journal of Farm Economics, XXVII(May, 1945), 303-14.

Von Neumann, John. "A Numerical Method to Determine Optimum Strategy," Naval Research Logistics Quarterly, I(June, 1954), 109-15.

Waugh, Frederick V. "The Minimum Cost of Dairy Feed," Journal of Farm Economics, XXXIII(August, 1951), 299-310.

Unpublished Material

Bureau of Ships, Department of the Navy. Daily Administrative Bulletin, 3 February, 1955.

Office of the Comptroller (Directorate of Management Analysis), Headquarters, U.S. Air Force. "Linear Programming." A paper read before the 2901st. Research and Development Unit, Army Reserve, by David Shepard, on 2 May, 1955, Washington, D.C. (Mimeographed.)

________. "New Applications of Linear Programming." A paper read before the national meeting of the Association for Computing Machinery, by Joseph V. Natrella, on 14-16 September, 1955, Philadelphia, Pa. (Mimeographed.)

________. The Application of Linear Programming Techniques to Air Force Problems. A non-technical discussion. AFAMA 3-4-52. Washington D.C., 17 December, 1954.

Other Sources

Bureau of Ships, Department of the Navy. Personal interviews with Commander R.H. Buck, USN, and Mr. Jack Smith, Electronics Division (code 990). December, 1955.

Office of the Comptroller (Directorate of Management Analysis), Headquarters, U.S. Air Force, Personal interviews with Mr. Joseph V. Natrella and Miss Natalie Coplan. December, 1955.

The George Washington University, Logistics Research Unit. Personal interview with Rear Admiral Ralph E. McShane, USN (Ret.). November, 1955.